The interior structure of black holes

and

the *strong cosmic censorship* conjecture in general relativity

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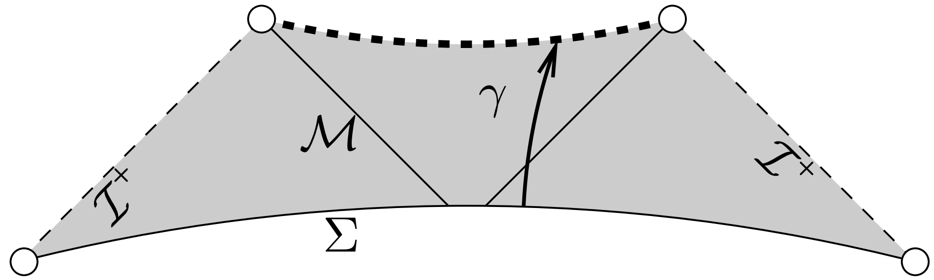
Outline

1. Schwarzschild, Kerr and the strong cosmic censorship conjecture
2. The blue-shift effect in linear theory
3. A fully non-linear "toy-model" under spherical symmetry
4. *Leaving toys behind*: null singularities for the vacuum Einstein equations <u>without symmetry</u>
5. What is left to be done?

1. Schwarzschild, Kerr and the strong cosmic censorship conjecture

Schwarzschild



The Schwarzschild spacetime $(\mathcal{M}^{3+1}, g)$ is a geodesically <u>in</u>complete black-hole solution of the *Einstein vacuum equations*

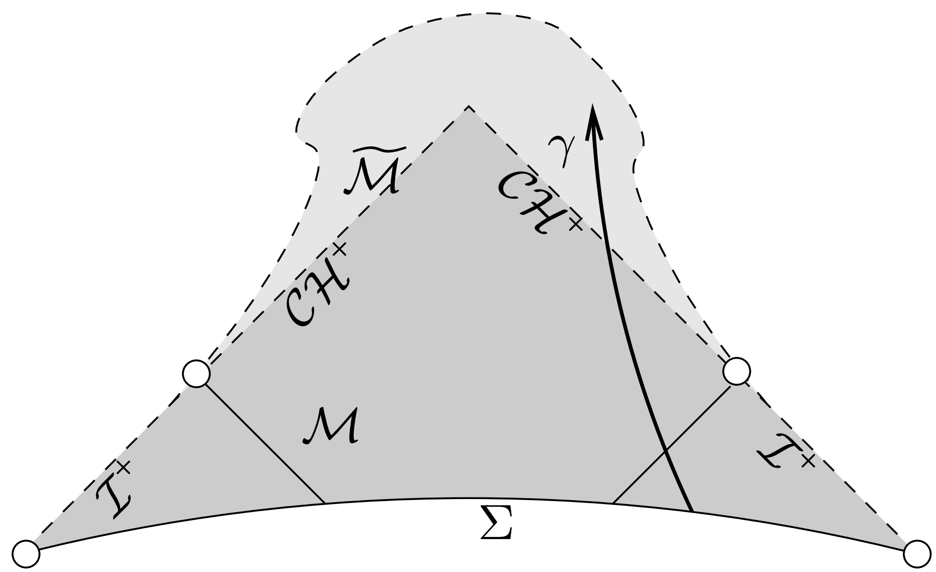
$$\mathrm{Ric}(g) = 0,$$

uniquely determined from initial data on a 2-ended asymptotically flat Cauchy hypersurface Σ. *Observers* γ *who enter the* **black hole region** *subsequently live only for <u>finite</u> proper time.*

Such observers γ are in fact "torn apart" by infinite tidal deformations. Spacetime terminates at a **spacelike singularity** across which it is *inextendible* as a manifold with continuous metric.

Kerr $0 < |a| < M$ and Reissner–Nordström $0 < Q < M$

Schwarzschild sits inside two larger families of solutions (Kerr and Reissner–Nordström) where the situation changes drastically:



The part of spacetime determined by initial data is smoothly extendible to a larger spacetime into which γ enters in finite time.

These extensions are non-unique. *What happens to* γ*?*

Strong cosmic censorship

Conjecture (Strong cosmic censorship, Penrose 1972)**.** *For generic asymptotically flat initial data* $(\Sigma, \bar{g}, K)$ *for the Einstein vacuum equations*

$$\mathrm{Ric}(g) = 0,$$

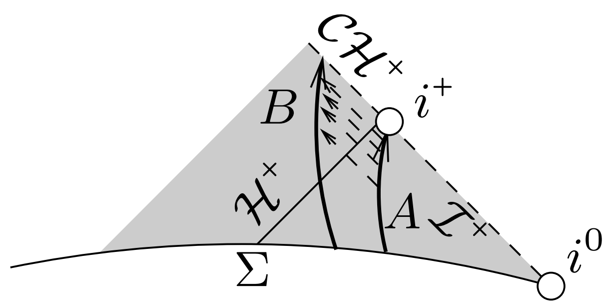
the solution spacetime $(\mathcal{M}, g)$ *determined by initial data* ***cannot be extended as a suitably regular Lorentzian manifold.***

One should think of this conjecture as a statement of *global uniqueness*, or, in more colloquial language:

> *"Generically, the future is uniquely determined by the present".*

Blue-shift instability (Penrose, 1968)

A possible mechanism for instability of the Cauchy horizon (and thus for the validity of the conjecture) is the celebrated blue-shift effect, first pointed out by Penrose:



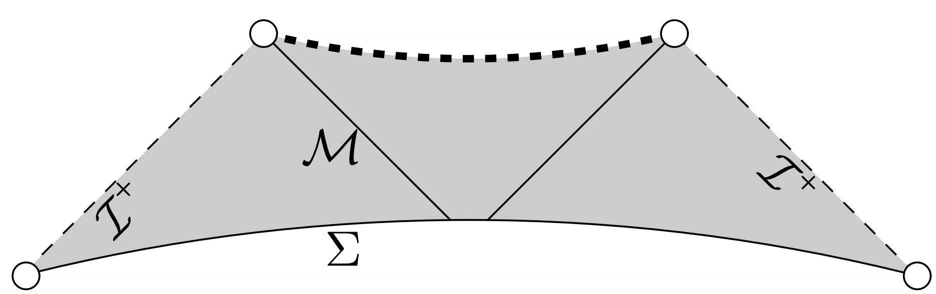
Penrose argued that this would cause solutions of the wave equation $\Box_g \psi = 0$ (thought of as a model for the *linearised* Einstein equations) to blow-up in some way on a fixed Reissner–Nordström background.

This suggests Cauchy horizon formation could indeed be an <u>unstable</u> phenomenon under evolution by the Einstein equations $\mathrm{Ric}(g) = 0$.

While linear perturbations as a matter of principle can at worst blow up *at the Cauchy horizon* $\mathcal{CH}^+$, in the full non-linear theory, one might expect that the non-linearities would kick in so as for blow-up to occur *before the Cauchy horizon has the chance to form*.

This motivates:

Conjecture (**Very** strong cosmic censorship)**.** *For generic vacuum asymptotically flat initial data* $(\Sigma, \bar{g}, K)$*, the maximal Cauchy development* $(\mathcal{M}, g)$ *is future inextendible as a Lorentzian manifold with continuous metric and the singularity can be naturally thought of as "spacelike".*



The blue-shift effect in linear theory

If we look at solutions of the wave equation $\Box_g \psi = 0$ on Kerr or Reissner–Nordström arising from *compactly supported* initial data, then the solutions decay outside the black hole and *a priori* this decay could compete with the blue-shift effect. We have, however:

Theorem 1 (M.D. 2003)**.** *In subextremal Reissner–Nordström, for sufficiently regular solutions of* $\Box\psi = 0$ *of initially compact support, then if the spherical mean* ψ_0 *satisfies*

$$|\partial_v \psi_0| \geq cv^{-4} \tag{1}$$

along the event horizon $\mathcal{H}^+$*, for some constant* $c > 0$ *and all sufficiently large* v*, then the energy measured by a local observer at the Cauchy horizon is indeed infinite:* $E[\psi] = \infty$*.*

The lower bound (1) is indeed suggested by approximations and numerics, cf. Price, Bicak, Gundlach–Price–Pullin, ...

The blow-up given by the above theorem, if it indeed occurs is, however, in a sense weak!

In particular, the amplitude $|\psi|$ of the solution remains bounded.

Theorem 2 (A. Franzen, 2013). *In subextremal Reissner–Nordström* **or Kerr** *with $M > Q \neq 0$ or $M > |a| \neq 0$, respectively, let ψ be a sufficiently regular solution of the wave equation. Then*

$$|\psi| \leq C$$

globally in the black hole interior up to and including $\mathcal{CH}^+$, to which ψ in fact extends continuously.

See upcoming results of Gajic for the extremal case $|a| = M$.

The proof of the above requires an upper bound for the decay rate of a scalar field along the event horizon $\mathcal{H}^+$ of a general subextremal Kerr metric $(0 < |a| < M)$.

This follows from very recent decay results of M.D.–Rodnianski–Shlapentokh-Rothman on the wave equation on exterior Kerr.

1. If one "naively" extrapolates the linear behaviour of $\Box\psi = 0$ to the non-linear $\mathrm{Ric}(g) = 0$, where we identify

$$\psi \sim g_{\mu\nu}$$

$$\partial\psi \sim \Gamma^{\lambda}_{\mu\nu}$$

this suggests that the <u>metric may extend continuously</u> to the Cauchy horizon whereas the **Christoffel symbols blow up**, failing to be square integrable, making the Cauchy horizon into a **weak null singularity**.

2. On the other hand, according to the "very strong" formulation of strong cosmic censorship, then the non-linearities of the Einstein equations should induce a more serious blow-up *earlier*, forming a **spacelike singularity**.

Which of the two scenario holds?

Fully non-linear toy-models under spherical symmetry

The Einstein–Maxwell–(real) scalar field model under spherical symmetry

The simplest toy model which allows for the study of this problem in spherical symmetry with a true wave-like degree of freedom is that of a self-gravitating *real-valued* scalar field in the presence of a self-gravitating electromagnetic field.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi (T^{\phi}_{\mu\nu} + T^{F}_{\mu\nu})$$

$$T^{\phi}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi$$

$$T^{F}_{\mu\nu} = \frac{1}{4\pi} (g^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta})$$

$$\Box_g \psi = 0, \qquad \nabla^\mu F_{\mu\nu} = 0, \qquad dF = 0$$

This generalises a model studied by CHRISTODOULOU with $F = 0$.

The system had been studied heuristically by Poisson–Israel and Ori, and then numerically by

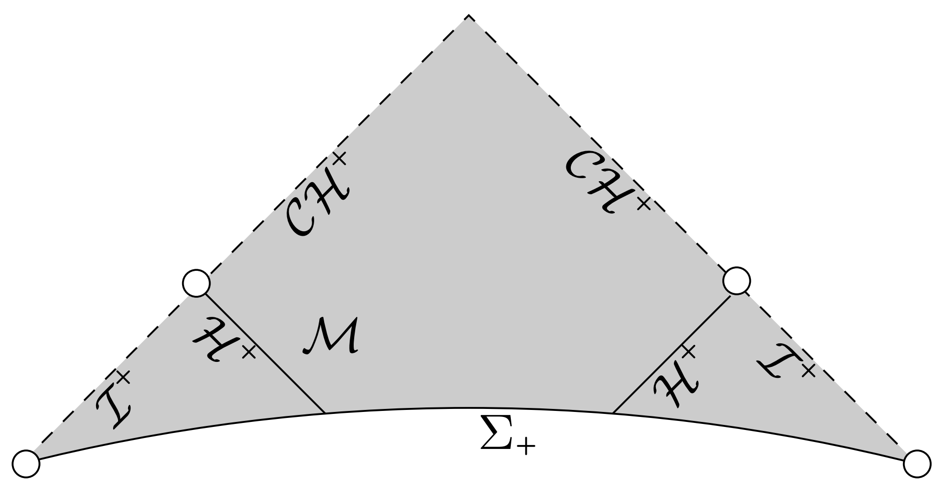
Gnedin–Gnedin 1993, Gundlach–Price–Pulin 1994, Bonano–Droz–Israel–Morsink 1995, Brady–Smith 1995, Burko 1997

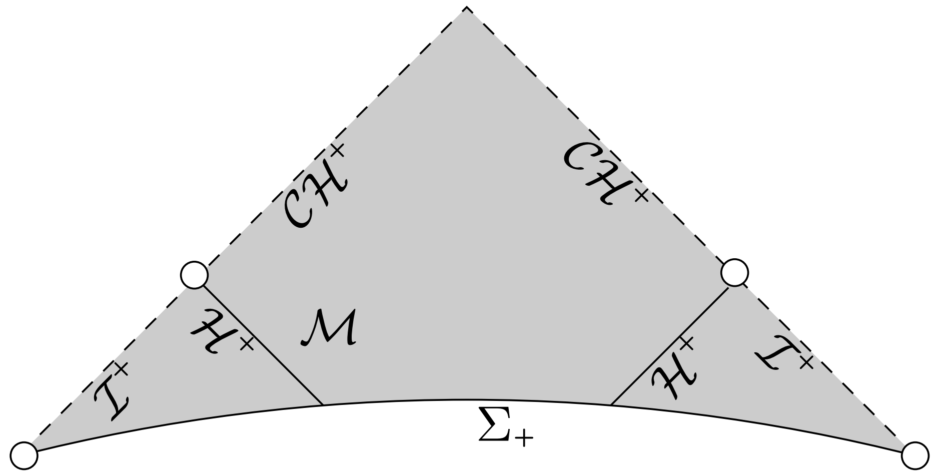
originally with conflicting results.

It turns out, however, that one can in fact mathematically *prove* essentially everything about the nonlinear dynamics of this model in a neighbourhood of the Reissner–Nordström solution.

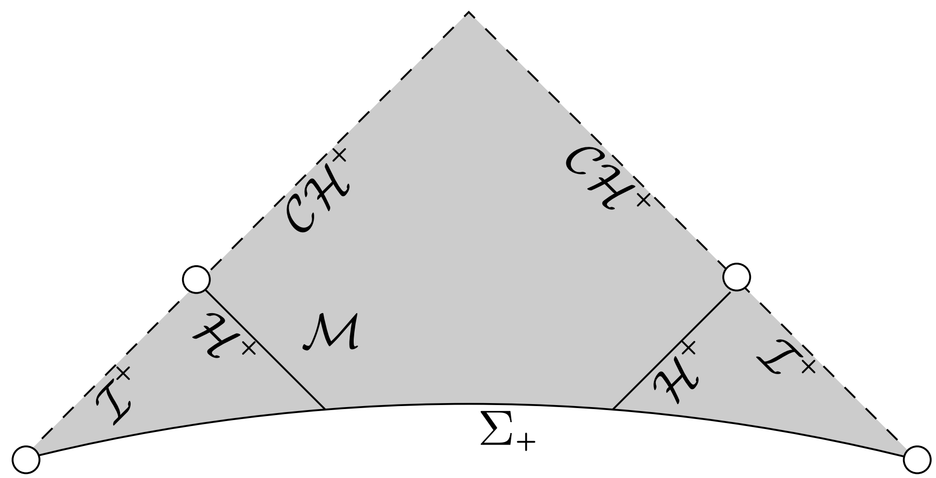
Theorem 3 (M.D. 2001, 2003, 2011). *Let* $(\mathcal{M}, g, \phi, F)$ *be the unique solution of the Einstein–Maxwell–real scalar field system evolving from* ***spherically symmetric*** *asymptotically flat two-ended data which are* <u>*sufficiently close*</u> *to exact subextremal Reissner–Nordström data with parameters* $0 < Q_{RN} < M_{RN}$.

1. Then, to the future of a Cauchy hypersurface Σ_+*, the Penrose diagram of* $(\mathcal{M}, g)$ *is again given by:*





2. Moreover, **the metric extends continuously** *beyond $\mathcal{CH}^+$ to a strictly larger Lorentzian manifold $(\widetilde{\mathcal{M}}, \widetilde{g})$, making $\mathcal{CH}^+$ a bifurcate null hypersurface in $\widetilde{\mathcal{M}}$. All future-incomplete causal geodesics in $\mathcal{M}$ extend to enter $\widetilde{\mathcal{M}}$. The scalar field ϕ extends to a continuous function on $\widetilde{\mathcal{M}}$.*



3. In addition, <u>*if*</u> *ϕ satisfies a pointwise* <u>*lower*</u> *bound on both components of the horizon $\mathcal{H}^+$ (cf. Theorem 1), then the* **Hawking mass diverges** *on all of $\mathcal{CH}^+$. In particular, $(\mathcal{M}, g)$ is future* **inextendible** *as a spacetime with square-integrable Christoffel symbols.*

Note that as in the proof of Theorem 2, an important ingredient in the proof of the above theorem are upper bounds on ϕ on $\mathcal{H}^+$, proven in M.D.–RODNIANSKI 2003.

The above theorem shows that in the context of the spherically symmetric toy-model, "naive" extrapolation of linear theory is indeed correct:

The blue-shift instability is **not** sufficiently strong to destroy the spacetime earlier in a spacelike singularity, but does give rise to a **weak null singularity** at the Cauchy horizon, across which the metric is however continuously extendibile.

Thus, in "toy-land", **very** strong cosmic censorship is **false**, but a weaker formulation requiring only inextendibility in the class of metrics with locally square integrable Christoffel symbols (a formulation due to CHRISTODOULOU) may still be true.

Is this indicative of the behaviour of the actual vacuum equations without symmetry, or is this just an artifact of the toy model?

Leaving toys behind:

Null singularities for the vacuum Einstein equations without symmetry

The first question one might ask is, can one construct **any** examples of weak null singularities for the vacuum?

This has recently been resolved in a remarkable new result of LUK.

LUK showed that if one "puts" in the expected singular profile of the initial shear $\hat{\chi}$ of a light cone (with affine parameter $\underline{u}$) singular as $\underline{u} \to 0$

$$|\hat{\chi}| \sim |\log(-\underline{u})|^{-p}|\underline{u}|^{-1}, \qquad p > 1$$

then, one can solve a characteristic initial value problem for the vacuum equations in a large enough region so as for this behaviour to propagate as a **weak null singular** boundary of spacetime.

No symmetry is required on initial data!

Here is the formal statement:

Theorem 4 (Luk). *Let us be given characteristic data for the Einstein vacuum equations* $\mathrm{Ric}(g) = 0$ *defined on a bifurcate null hypersurface* $\mathcal{N}^{\mathrm{out}} \cup \mathcal{N}^{\mathrm{in}}$*, where* $\mathcal{N}^{\mathrm{out}}$ *is parameterised by affine parameter* $\underline{u} \in [\underline{u}^*, 0)$*, and the data are regular on* $\mathcal{N}^{\mathrm{in}}$ *while singular on* $\mathcal{N}^{\mathrm{out}}$*, according to*

$$|\hat{\chi}| \sim |\log(-\underline{u})|^{-p} |\underline{u}|^{-1}, \tag{2}$$

for appropriate $p > 1$*.*

Then the solution exists in a region foliated by a double null foliation with level sets u*,* $\bar{u}$ *covering the region* $u^* \leq u < 0$*,* $\underline{u}^* \leq \underline{u} < 0$ *for* $\underline{u}^*$ *as above and sufficiently small* u^**, and the bound* (2) *propagates. The spacetime is continuously extendible beyond* $\underline{u} = 0$*, but the Christoffel symbols fail to be square integrable in this extension.*

The above theorem tell us that weak null singularities for the vacuum are a reality!

It does not yet tell us, however, that null boundaries form inside generic black holes.

Combining Luk's methods with the intuition and methods derived from the spherically symmetric toy-model, we have recently obtained the following definitive result concerning vacuum black hole interiors without symmetry, which I will announce here:

Here is the formal statement:

Theorem 5 (M.D.–Luk, to appear)**.** *Let us be given characteristic initial data for the Einstein vacuum equations,* ***with no symmetry assumed****, defined on two intersecting future-affine complete null hypersurfaces* $\mathcal{H}^+_A \cup \mathcal{H}^+_B$*, such that, along each, the data are near and in fact asymptote to (at a sufficiently fast inverse polynomial rate) event-horizon data of a subextremal Kerr with* $a \neq 0$*.*

Then the solution exists up to **a bifurcate Cauchy horizon** *(just as in Kerr!) beyond which the metric extends continuously (but at which the Christoffel symbols may blow up, failing even to be square integrable).*

In simpler language, what is proven is the following:

If the <u>exterior</u> regions of the Kerr spacetime are indeed dynamically stable (as is universally believed–but not yet shown!), then **so is the entirety of its Cauchy horizon** *as to its null character and the continuous extendibility of the metric beyond it.*

The result still would allow the Christoffel symbols to blow up failing even to be square integrable at the Cauchy horizon, in which case the horizon would represent a weak null singularity.

More generally, the above theorem in fact implies the following:

Any dynamic spacetime settling down to Kerr in its *exterior* region at inverse polynomial rates will necessarily have a piece of ***non-empty Cauchy horizon*** in its interior, possibly singular, but across which the metric still extends continuously.

What is left to be done?

Conjecture 1 (Stability of the Kerr exterior). *Small perturbations of Kerr initial data on a Cauchy hypersurface indeed form an event horizon outside of which the solution settles down to a nearby Kerr solution at a sufficiently fast inverse polynomial rate.*

If the above conjecture is true, then the statement of our theorem applies to all spacetimes arising from sufficiently small perturbations of Kerr initial data on a spacelike hypersurface, showing that such spacetimes can be extended as a continuous Lorentzian metric across a bifurcate Cauchy horizon.

In particular, a corollary of the above and our theorem would be

Corollary. *Very strong cosmic censorship is false.*

Conjecture 2. *For generic initial data sufficiently close to Kerr, the resulting Cauchy horizon is indeed (globally) singular in the sense that any C^0 extension $\widetilde{\mathcal{M}}$ as above will fail to have L^2 Christoffel symbols in a neighbourhood of any point of $\partial\mathcal{M}$.*

This corresponds to the inextendibility statement which was conditionally shown in the spherically symmetric toy model.

Though not sufficient to show that macroscopic observers are "torn apart" in the sense of a naive Jacobi field calculation, this would ensure that the boundary of spacetime is singular enough so that one cannot extend the spacetime as a *weak solution* to the Einstein equations.

In particular, a corollary of the above and our theorem would be

Corollary. CHRISTODOULOU*'s formulation of strong cosmic censorship is true* ***in a neighbourhood of the Kerr family****.*

What is the global picture when one does **not** begin with initial data sufficiently close to Kerr data?

Is there, in addition to the null component, in general also a spacelike portion of the singularity?

Or does this null piece necessarily "close up" before such a spacelike component can occur?